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Inequalities for analytic functions with the derivative in H^1

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Abstract

It is proved that if $f' \in H^1$, then

$$\int\limits_T \frac{\left|f(\zeta\eta) - f(\overline{\zeta}\eta)\right|}{|1-\zeta|} dm(\zeta) = \frac{1}{\pi} \int\limits_0^\pi \frac{\left|f(e^{i(\theta+t)}) - f(e^{i(\theta-t)})\right|}{2\sin(t\backslash 2)} \leq \pi \left\|f'\right\|_{H^1},\ \eta = e^{i\theta}.$$

1 Introduction

Let A be the class of all functions analytic in the unit disc $D = \{\zeta : \ |\zeta| < 1\}$, $m(\zeta)$ - normalized Lebesgue measure on the circle $T = \{\zeta : \ |\zeta| = 1\}$. Let H^p $(1 \leq p \leq \infty)$ is the space of all functions analytic in D and satisfying

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \left(\int_T |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} < \infty, \qquad 1 \leq p < \infty,$$

$$\|f\|_{H^\infty} = \sup_{z \in D} \ |f(z)| < \infty, \qquad p = \infty.$$

For $f \in A,\ \ f' \in H^1$ it follows the Hardy inequality [1, 104-105]:

$$\sum_{k \geq 1} \left|\hat{f}(k)\right| \ \leq \ \pi \left\|f'\right\|_{H^1}.$$

In this paper we shall prove an inequality - integrated analogue of the Hardy inequality and as the application we shall give simplified proof of the theorem of S. A. Vinogradov for the bounded Toeplitz operators on H^∞ [2].

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2 Main results

Theorem 1. *If $f \in A$ and $f' \in H^1$ then*

$$\int_T \frac{\left|f(\zeta\eta) - f(\overline{\zeta}\eta)\right|}{|1-\zeta|} dm(\zeta) \;\le\; \pi \, \|f'\|_{H^1} \,, \quad \eta \in T.$$

Proof. Let $f' \in H^1$, $\eta \in T$. Then

$$\int_T \frac{\left|f(\zeta\eta) - f(\overline{\zeta}\eta)\right|}{|1-\zeta|} dm(\zeta) \;\le\; \lim_{r \to 1-0} \int_T \frac{\left|f(r\zeta\eta) - f(r\overline{\zeta}\eta)\right|}{|1-\zeta|} dm(\zeta) \,. \qquad (1)$$

Since $f' \in H^1$, integrating in parts and applying the Cauchy theorem, we obtain

$$\tfrac{1}{2\pi i} \int_T f'(\xi) \, ln \, |1 - \xi\overline{z}|^2 \, d\xi \quad =$$

$$\tfrac{1}{2\pi i} \int_T f'(\xi) \, ln(\, 1 - \xi\overline{z} \,) d\xi \; + \tfrac{1}{2\pi i} \int_T f'(\xi) \, ln(\, 1 - \overline{\xi} z \,) d\xi =$$

$$\tfrac{1}{2\pi i} \int_T f'(\xi) \, ln(\, 1 - \xi\overline{z} \,) d\xi + \tfrac{1}{2\pi i} \int_T f'(\xi) \, ln(\, \xi - z \,) d\xi - \tfrac{1}{2\pi i} \int_T f'(\xi) \, ln \, \xi \, d\xi =$$

$$\tfrac{1}{2\pi i} \int_T f(\xi) (\, \tfrac{\overline{z}}{1 - \xi\overline{z}} \; - \; \tfrac{1}{\xi - z} \; + \; \tfrac{1}{\xi} \,) \, d\xi \;=\; f(0) - f(z) \quad \Rightarrow$$

$$f(z) = f(0) - \frac{1}{2\pi i} \int_T f'(\xi) \, ln \, |1 - \xi\overline{z}|^2 \, d\xi.$$

Using last equality for $f(z\eta)$:

$$f(z\eta) = f(0) - \frac{1}{2\pi i} \int_T f'(\xi\eta) \, ln \, |1 - \xi\overline{z}|^2 \, d\xi,$$

we shall have

$$\left|f(r\zeta\eta) - f(r\overline{\zeta}\eta)\right| = \left| \frac{1}{2\pi i} \int_T f'(\xi\eta) \, ln \left| \frac{1 - \xi r\zeta}{1 - \xi r\overline{\overline{\zeta}}} \right|^2 d\xi \right| \le$$

$$\le \int_T |f'(\xi\eta)| \; \left| ln \left| \frac{1 - \xi r\zeta}{1 - \xi r\overline{\overline{\zeta}}} \right|^2 \right| dm(\xi) =$$

$$= \int_T |f'(\xi\eta)| \; \left| ln \frac{(1-r)^2 + r \, |1 - \xi\zeta|^2}{(1-r)^2 + r \, \left|1 - \xi\overline{\zeta}\right|^2} \right| dm(\xi) \le$$

$$\leq \int_T |f'(\xi\eta)| \quad \left| ln \left| \frac{1-\xi\zeta}{1-\xi\overline{\zeta}} \right|^2 \right| dm(\xi).$$

Further from (1) follows

$$\int_T \frac{|f(\zeta\eta) - f(\overline{\zeta}\eta)|}{|1-\zeta|} dm(\zeta) \leq 2 \int_T \int_T \frac{|f'(\xi\eta)|}{|1-\zeta|} \left| ln \left| \frac{1-\xi\zeta}{1-\xi\overline{\zeta}} \right| \right| dm(\xi) dm(\zeta) \leq$$

$$\leq \|f'\|_{H^1} \sup \left\{ 2 \int_T \left| ln \left| \frac{1-\xi\zeta}{1-\xi\overline{\zeta}} \right| \right| \frac{dm(\zeta)}{|1-\zeta|} \quad : \quad \xi \in T \right\}.$$

For end the proof is necessary only to estimate the integral

$$I(\xi) = 2 \int_T \left| ln \left| \frac{1-\xi\zeta}{1-\xi\overline{\zeta}} \right| \right| \frac{dm(\zeta)}{|1-\zeta|}, \quad \xi \in T.$$

We have

$$I(e^{i\theta}) = 2.\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left| ln \left| \frac{1-e^{i(\theta+t)}}{1-e^{i(\theta-t)}} \right| \right| \frac{dt}{|1-e^{it}|} \quad =$$

$$= \frac{1}{\pi} \int\limits_{0}^{\pi} \left| ln \left| \frac{\sin((\theta+t)/2)}{\sin((\theta-t)/2)} \right| \right| \frac{dt}{\sin(t/2)} \quad =$$

$$= \frac{1}{\pi} \int\limits_{0}^{\pi} \left| ln \left| \frac{tg(\theta/2) + tg(t/2)}{tg(\theta/2) - tg(t/2)} \right| \right| \frac{dt}{\sin(t/2)} = \frac{2}{\pi} \int\limits_{0}^{\infty} \left| ln \left| \frac{tg(\theta/2) + y}{tg(\theta/2) - y} \right| \right| \frac{dy}{y\sqrt{1+y^2}} \leq$$

$$\frac{2}{\pi} \int\limits_{0}^{\infty} \left| ln \left| \frac{tg(\theta/2) + y}{tg(\theta/2) - y} \right| \right| \frac{dy}{y} \leq \frac{2}{\pi} \int\limits_{0}^{\infty} \left| ln \left| \frac{1+x}{1-x} \right| \right| \frac{dx}{x} = \frac{2}{\pi} \int\limits_{0}^{1} \ln \left(\frac{1+x}{1-x} \right) \frac{dx}{x} = \pi.$$

Application of Theorem 1.

For $f \in H^1$ we denote by T_f the Toeplitz operator on H^∞ , defined by

$$T_f h = \int_T \frac{\overline{f}(\zeta) h(\zeta)}{1-\overline{\zeta} z} dm(\zeta), \qquad h \in H^\infty.$$

In [2] S. A. Vinogradov proves that if $f' \in H^1$, then the Toeplitz operator T_f is bounded on H^∞ .

As the application of Theorem 1 we shall give a simplified proof of the theorem of S. A. Vinogradov and estimation for $\|T_f\|_{H^\infty}$.

Theorem 2. *If* $f \in A$ *and* $f' \in H^1$ *then*

$$\|T_f\|_{H^\infty} \le \|f\|_{H^\infty} + \pi \|f'\|_{H^1}.$$

Proof.

$$\|T_f\|_{H^\infty} = \sup\left\{ \lim_{r\to 1-0} \left| \int_T \frac{\overline{f}(\zeta)h(\zeta)}{1-\overline{\zeta} r\eta} dm(\zeta) \right| : \eta \in T, \|h\|_{H^\infty} \le 1 \right\} =$$

$$= \sup\left\{ \lim_{r\to 1-0} \left| \int_T \frac{\overline{f}(\zeta\eta)h(\zeta\eta)}{1-r\overline{\zeta}} dm(\zeta) \right| : \eta \in T, \|h\|_{H^\infty} \le 1 \right\} \le$$

$$\le \sup\left\{ \lim_{r\to 1-0} \left| \int_T \frac{\overline{f}(\zeta\eta)-\overline{f}(\overline{\zeta}\eta)}{1-r\overline{\zeta}} h(\zeta\eta) dm(\zeta) \right| : \eta \in T, \|h\|_{H^\infty} \le 1 \right\} + \|f\|_{H^\infty}.$$

We used, that $g(z) = \overline{f}(\overline{z}\eta) \in H^\infty$ and

$$\left| \int_T \frac{\overline{f}(\overline{\zeta}\eta)h(\zeta\eta)}{1-r\overline{\zeta}} dm(\zeta) \right| \le \|f\|_{H^\infty} \|h\|_{H^\infty}.$$

Further from Theorem 1 follows

$$\|T_f\|_{H^\infty} \le \sup\left\{ \left| \int_T \frac{|f(\zeta\eta)-f(\overline{\zeta}\eta)|}{|1-\zeta|} dm(\varsigma) \right| : \eta \in T \right\} + \|f\|_{H^\infty} \le$$

$$\le \ \pi \|f'\|_{H^1} + \|f\|_{H^\infty}.$$

References

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